MTH 304 Midterm Solutions

1. Let A be a set; $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in J}$ be an indexed family of topological spaces; and let $\{f_\alpha\}_{\alpha\in J}$ be the indexed family of functions $f_\alpha : A \to X_\alpha$.

   (a) Show that there is a unique coarsest topology $\mathcal{T}$ on A with respect to which each f_α is continuous.

   (b) Show that the collection

   $$S = \{f_\beta^{-1}(U_\beta) \mid \beta \in J \text{ and } U_\beta \in \mathcal{T}_\beta\}$$

   forms a subbasis for $\mathcal{T}$.

   (c) Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be defined by

   $$f(a) = (f_\alpha(a))_{\alpha\in J}.$$

   Show that if $U \in \mathcal{T}$, then $f(U)$ is an open subset of $f(A)$.

Solution. (a) & (b) Consider the collection

$$S = \{f_\beta^{-1}(U_\beta) \mid \beta \in J \text{ and } U_\beta \in \mathcal{T}_\beta\}.$$

Clearly, the union of elements in S equals A, and hence S forms a subbasis for a topology $\mathcal{T}$ on X. By definition, each function $f_\alpha : A \to X_\alpha$ will be continuous under $\mathcal{T}$.

Suppose that $\mathcal{T}'$ is another topology on A in which each f_α is continuous. Then $f_\beta^{-1}(U_\beta) \in \mathcal{T}'$, for all $\beta \in J$ and $U_\beta \in \mathcal{T}_\beta$. Since $\mathcal{T}'$ is closed under arbitrary unions and finite intersections, we have that $\mathcal{T} \subset \mathcal{T}'$. Therefore, it follows that $\mathcal{T}$ is the unique coarsest topology under which each f_α is continuous.

(c) It suffices to consider the case when U is a basic open set (why?). Any basic open set $U \in \mathcal{T}$ must be of the form

$$\bigcap_{k=1}^{n} f_{\beta_k}^{-1}(U_{\beta_k}) = \bigcap_{k=1}^{n} (\pi_{\beta_k} \circ f)^{-1}(U_{\beta_k}).$$

We claim that

$$f(U) = \bigcap_{k=1}^{n} \pi_{\beta_k}^{-1}(U_{\beta_k}) \cap f(A). \qquad (*)$$

If $x = (x_\alpha) \in \bigcap_{k=1}^{n} \pi_{\beta_k}^{-1}(U_{\beta_k}) \cap f(A)$, then we have

$$x \in \pi_{\beta_k}^{-1}(U_{\beta_k}), \text{ for } 1 \leq k \leq n \text{ and } x \in f(A). \qquad (**)$$

Since $x \in f(A)$, there exists $a \in A$ such that $f(a) = x$. To show that $x \in f(U)$, it suffices to show that

$$a \in f_{\beta_k}^{-1}(U_{\beta_k}), \text{ for } 1 \leq k \leq n,$$

which is equivalent to showing that

$$f_{\beta_k}(a) \in U_{\beta_k}, \text{ for } 1 \leq k \leq n,$$

but this follows from (**), as $f_{\beta_k}(a) = x_{\beta_k}$. By reversing this argument, we obtain (*). Finally, since $\cap_{k=1}^{n} \pi_{\beta_k}^{-1}(U_{\beta_k})$ is a basic open set in the product topology in $\prod_{\alpha \in J} X_\alpha$, we have that $f(U)$ is an open subset of $f(A)$.

2. The subset

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

of $[0,1]$ is called the *Cantor Set*.

(a) Show that C is closed and bounded. (Note: This shows that C is compact.)

(b) Show that C is totally disconnected

Solution. (a) First, note that

$$C = \bigcap_{n=1}^{\infty} C_n, \text{ where}$$

$$C_n = C_{n-1} - \bigcup_{k=1}^{\infty} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right).$$

Each C_n is closed, being a finite intersection of closed intervals. As C is an infinite intersection of the closed setsC_n, it is closed. The boundedness of C follows from the fact that $C \subset [0,1]$.

(b) This is a standard exercise from a first course in real analysis. Let x, y be distinct points in C. Then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{3^{n-1}} < |x - y|.$$

Since each C_n is a disjoint union of closed intervals of length $\frac{1}{3^{n-1}}$, we have that $x, y \in C_n$. Furthermore, x and y must be contained in two of the distinct (and disjoint) closed intervals that make up C_n. If J be the closed interval in C_n containing x, then

$$x \in C \cap J \text{ and } y \in C \setminus J,$$

and these are closed sets that separate C. Therefore, C is totally disconnected (why?).

3. Consider the unit circle S^1 centered at the origin. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exists a point $x \in S^1$ such that $f(x) = f(-x)$. [Hint: Consider the map $g(x) = f(x) - f(-x)$, and use the connectedness of S^1.]

   Solution. As suggested, define a new function

$$g : S^1 \to \mathbb{R} : x \mapsto f(x) - f(-x).$$

   Then g is a continuous map from a connected space into $\mathbb{R}$. Moreover, since $g(-x) = -g(x)$, for all $x \in S^1$, the Intermediate Value Theorem would imply that there exists $y \in S^1$ such that $g(y) = 0$ (why?). This would imply that

$$f(y) - f(-y) = 0,$$

   and the result follows.

4. A topological space G is called a *topological group* if G forms a group under an operation $\cdot$ such that

$$(g, h) \mapsto g \cdot h, \forall g, h \in G$$

   and

$$g \mapsto g^{-1}, \forall g \in G$$

   are continuous maps. Let G be a topological group and let H be a subgroup of G.

(a) Show that the quotient map $G \to G/H$ is an open map.

(b) Show that if $H \trianglelefteq G$, then G/H is a topological group.

(c) Show that $\mathbb{R}/\mathbb{Z}$ is a topological group. Describe this group.

Solution. (a) For a fixed $g \in G$, consider the $\phi_g : G \to G$ defined by

$$\phi_g(h) = hg, \forall h \in G.$$

Since $(\phi_g)^{-1} = \phi_{g^{-1}}$, it follows that ϕ_g is a bijection. Moreover, as multiplication and inversion within the group G are continuous maps, it follows that ϕ_g is a homeomorphism. Consequently, if U is an open set in G, then

$$\phi_g(U) = Ug = \{ug \mid u \in U\}$$

is also an open set of U. Now let $q : G \to G/H$ denote the quotient map. Then we have

$$q(U) = UH = \bigcup_{g \in H} Ug,$$

which shows that $q(U)$ is open.

(b) If $H \trianglelefteq G$, then we know (from MTH 301) that G/H is a group. Furthermore, G/H inherits a natural quotient topology from G, as it is a collection of disjoint left cosets of H in G, more precisely, the quotient topology on G/H is defined by

$$G/H = G/\sim_H, \text{ where } x \sim_H y \iff xy^{-1} \in H, \forall x, y \in G.$$

It remains to show that the maps

$$\begin{array}{c} (gH, g'H) \mapsto gg'H \text{ and} \\ gH \mapsto g^{-1}H \end{array} \tag{∇}$$

are continuous maps. We will fix the following notation for the remaining part of this argument

$$\begin{array}{rlrl} g' \overset{\varphi_g}{\longmapsto} gg' & & g'H \overset{\bar{\varphi}_g}{\longmapsto} gg'H \\ (g, g') \overset{\mu}{\mapsto} gg' & & (gH, g'H) \overset{\bar{\mu}}{\mapsto} gg'H \\ g \overset{\iota}{\mapsto} g^{-1} & & gH \overset{\bar{\iota}}{\mapsto} g^{-1}H \end{array}$$

For establishing (∇), it suffices to show that the maps $\bar{\varphi}_g$ for all $g \in G$, and $\bar{\iota}$ are continuous maps (why?). First, note that $\varphi_g \in \mathrm{Aut}(G)$ (from MTH 301), and further using an argument analogous to (a), we can conclude that ϕ_g is a self homeomorphism of G. In a similar manner, we can also infer that ι is also a self homeomorphism of G. Using these observations and the fact that q is an open map, we obtain the following commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\varphi_g} & G \\ \Big\downarrow q & & q \Big\downarrow \\ G/H & \xrightarrow{\bar{\varphi}_g} & G/H \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\mu} & G \\ \Big\downarrow q & & q \Big\downarrow \\ G/H & \xrightarrow{\bar{\mu}} & G/H \end{array}$$

From these diagrams, we can infer that (∇) holds true.

(c) Since $(\mathbb{Z}, +) \trianglelefteq (\mathbb{R}, +)$, it follows from (b) that $\mathbb{R}/\mathbb{Z}$ is a topological group. From class, we know that (why?)

$$\mathbb{R}/\mathbb{Z} \approx [0,1]/0 \sim 1 \approx S^1.$$

5. Let X be the quotient space obtained from $\mathbb{R} \times \{0,1\}$ by identifying $(x, 0)$ with $(x, 1)$ for every number $x \in \mathbb{R}$ with $|x| > 1$.

   (a) Does X satisfy the T_1 axiom?

   (b) Is X Hausdorff?

Solution. (a) We claim that X is a T_1 space. To show this, consider points x and y be distinct points in X. If they differ in their first coordinate, then they can be separated by open sets (using the Hausdorff property). If x and y are points of the form $[r \times 0]$ and $[r \times 1]$ respectively with $|r| < 1$, then they can be separated by disjoint open neighborhoods which are the images of the sets $U \times \{0\}$ and $U \times \{1\}$, for some $U \subset (1, 1)$. When $\{x, y\} = \{[1 \times 0], [1 \times 1]\}$ then each of x, y has a neighborhood, not containing the other (why?) Finally, a similar argument holds true for the case when for $\{x, y\} = \{[1 \times 0], [1 \times 1]\}$, and our claim follows.

(b) We show that X cannot be Hausdorff, by proving that the points $[1 \times 0]$ and $[1 \times 1]$ are cannot be separated by open sets. Every neighborhood of $[1 \times 1]$ must contain a set whose preimage in $\mathbb{R} \times \{0,1\}$ contains an interval $(1-\epsilon, 1+\epsilon) \times \{1\}$ around 1×1 (why?), and since it is saturated, it must also contain $(1, 1+\epsilon) \times \{1\}$. Neighborhoods of the point $[1 \times 0]$ must also contain a similar set. Two such neighborhoods will always have points in common (why?).

6. **(Bonus)** Let S^2 be the unit two sphere in $\mathbb{R}^3$ centered at the origin. Consider an equivalence relation $\sim$ on $X = S^2 \times [0,1]$ defined by

$$(x,i) \sim (y,j) \iff x = y,\ i = 0,\ \text{and } j = 1.$$

Show that $X/\sim$ is a compact and connected 3-manifold that is imbeddable in $\mathbb{R}^5$.

Solution. For every point $x \in S^2$, the set $\{x\} \times [0,1]$ is a homeomorphic to $[0,1]$. Since we know from class that $[0,1]/0 \sim 1 \approx S^1$, it immediately follows that each for each $x \in S^1$,

$$\{x\} \times [0,1]/(x,0) \sim (x,1) \approx S^1.$$

Hence, we have that (why?)

$$X/\sim \approx S^2 \times S^1.$$

Since $X/\sim$ is a product compact spaces, it is compact. Furthermore, since $X/\sim$ is continuous image (under the quotient map) of a connected space X (why?), $X/\sim$ will be connected.

Finally, since S^1 embeds into $\mathbb{R}^2$ and S^2 embeds into $\mathbb{R}^3$ (by identifying them respectively with the unit spheres in $\mathbb{R}^2$ and $\mathbb{R}^3$ centered at origin), we can see that $X/\sim$ embeds into $\mathbb{R}^5$ (why?) In fact, smallest positive integer n such that $X/\sim$ is imbedable in $\mathbb{R}^n$ is 5 (why?).